The Ferromagnetic Potts model under an external magnetic field: an exact renormalization group approach

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The q-state ferromagnetic Potts model under a non-zero magnetic field coupled with the 0th Potts state was investigated by an exact real-space renormalization group approach. The model was defined on a family of diamond hierarchical lattices of several fractal dimensions d_F. On these lattices, the renormalization group transformations became exact for such a model when a correlation coupling that singles out the 0th Potts state was included in the Hamiltonian. The rich criticality presented by the model with $q = 3$ and $d_F = 2$ was fully analyzed. Apart from the Potts criticality for the zero field, an Ising-like phase transition was found whenever the system was submitted to a strong reverse magnetic field. Unusual characteristics such as cusps and dimensional reduction were observed on the critical surface.

I. INTRODUCTION

The q-state Potts model is one of the most studied models in statistical mechanics due to its wide theoretical interest and practical applications [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. The pure ferromagnetic version of this model in a zero field is known to exhibit a phase transition from the high-temperature disordered paramagnetic phase to the low-temperature ferromagnetic for $q > 1$ and Bravais lattices with dimensions $d \geq 2$ [2, 7].

From a theoretical viewpoint, the q-state Potts model in the absence of an external magnetic field and defined on many kinds of Bravais and fractal lattices, has been extensively studied by mean-field-like and numerical approaches for ferromagnetic, anti-ferromagnetic and disordered couplings. However, few exact results have been obtained for such models [7, 13, 14, 15]. Exact results are very important as a guide for gaining an insight into many complicated points of the phase diagram.

The q-state Potts model under an external magnetic field in its turn, due to its complexity, has been studied by only a few authors [16, 17, 18, 19] and several aspects of its physical behavior have remained poorly established.

This paper has studied the role of the magnetic field on the physical properties of the ferromagnetic q-state Potts model defined on a family of fractal lattices, called diamond hierarchical lattices (DHL). The zero-field version of this model was previously studied by the authors [13]. One important issue for the study of spin models on such lattices arises from the scale invariance property which is presented and that leads to exact and tractable solutions. Generally speaking, in qualitative terms, they compare favorably to similar ones obtained by other approximate methods. Indeed, the study of spin models on such lattices can be viewed as the counterpart of the real-space renormalization group approximation for the corresponding models defined on Bravais lattices [20, 21]. Therefore, they provide an alternative framework for studying the critical behavior of more realistic systems whenever approaches considering translational invariant lattices lead to untractable procedures.

To benefit from the scale invariance property of the DHL, an exact real-space renormalization group (RG) scheme was applied in order to study the q-state Potts model on such lattices. Closed renormalization transformations in the coupling parameter space were achieved when a 0-state correlation coupling interaction was introduced into the Hamiltonian, as well as the constants of the magnetic field and ferromagnetic coupling. Under such 0-state correlation interaction pairs of spins in the 0th Potts state, aligned with the magnetic field, are singled out. The particular model with $q = 3$ Potts states defined on a DHL with a scaling factor of two and fractal dimension of $d_F = 2$ was exhaustively studied. The full flow diagram of the closed RG equations in the three dimensional parameter space was obtained exhibiting the stable fixed points associated with the zero-field ferromagnetic and paramagnetic Potts phases, as well as those associated with an Ising-like phase induced by a strong reverse magnetic field. The critical surface delineating the basin of attraction of the Ising-like fixed point was sketched, revealing unusual characteristics such as cusps and dimensional reduction.

This paper is organized as follows: the Hamiltonian model and the general renormalization equations for the model with q Potts states defined on a general DHL with an arbitrary fractal dimension are presented and discussed in section II, and its deduction is displayed in the
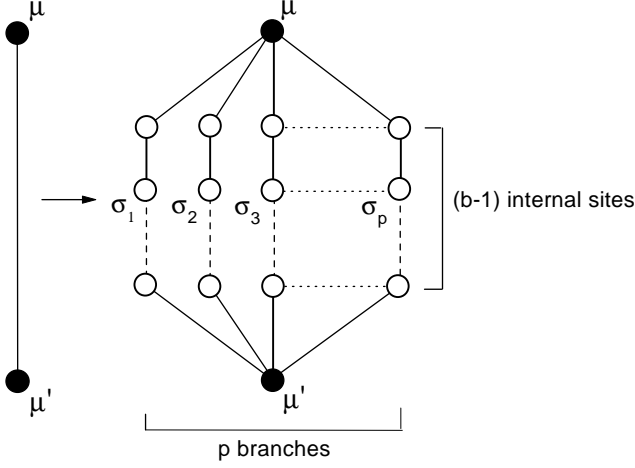
FIG. 1: Inflation process for a general diamond hierarchical lattice with scaling factor b and p branches.

Appendix. The particular model for $q = 3$ and $d_F = 2$ is studied in section III, the corresponding renormalization equations set in subsection III A, while subsection III B is devoted to presenting the critical points in the coupling constant parameter space, and to analyzing the full renormalization flow and unusual characteristics appearing on the critical surface. Finally, the discussion is presented and the conclusions are summarized in section IV.

II. THE HAMILTONIAN MODEL AND METHODOLOGY

DHL lattices were constructed commencing from a basic unit (called first generation or hierarchy) and replaced all of its single bonds with the basic unit itself to build the second generation. Such a replacing procedure is sketched in Figure 1 for the general DHL, with a scaling factor b and p connecting branches, hereafter referred to as (b, p)-DHL. This inflation procedure is recursively applied leading to a two-root lattice with the fractal dimension, the number of sites and the number of bonds given by $d_F = 1 + \log p / \log b$, $N_S^n = 2 + (b-1)p[(bp)^n - 1]/(bp - 1)$ and $N_B^n = (bp)^n$, respectively, with n as the number of generations or hierarchies.

The general q-state Potts Hamiltonian $\mathcal{H}^{(n)}$ on a n-generation (b, p)-DHL can be written as:

$$-\beta \mathcal{H}^{(n)} = \sum_{<i,j>} q K_{ij}^{(n)} \delta_{\sigma_i \sigma_j} +$$
$$+ \sum_i q H_i^{(n)} \delta_{\sigma_i \, 0} + \sum_{<i,j>} q L_{ij}^{(n)} \delta_{\sigma_i \, 0} \delta_{\sigma_j \, 0}, \quad (1)$$

where $\beta = 1/kT$, T is the absolute temperature and $\delta_{\sigma, \sigma'}$ is the Kronecker delta function. The energy of the ferro-

magnetic interaction coupling the nearest-neighbor spins σ's ($\sigma = 0, 1, 2, \ldots q$) is described by the first term of the equation (1), while the energy of the interaction of the external magnetic field aligned to the 0th state is given by the second term. Finally, the above mentioned correlation coupling interaction, which bolsters the energy of the pairs of spins in the 0th-state is accounted for by the third term.

The Hamiltonian function for the pure (homogeneous) ferromagnetic Potts model obtained from equation (1) when all coupling constant interactions are considered independent of the site position, that is $K_{ij}^{(n)} \equiv K^{(n)}$, $L_{ij}^{(n)} \equiv L^{(n)}$, and $H_{ij}^{(n)} \equiv H^{(n)}$, is simplified as

$$-\beta \mathcal{H}^{(n)} = q K^{(n)} \sum_{<i,j>} \delta_{\sigma_i \sigma_j} +$$
$$+ q H^{(n)} \sum_i \delta_{\sigma_i \, 0} + q L^{(n)} \sum_{<i,j>} \delta_{\sigma_i \, 0} \delta_{\sigma_j \, ,0} \quad (2)$$

The pure q-state Potts model, defined on the (b, p)-DHL with n generations, can be exactly transformed by renormalization in an equivalent model defined on the (b, p)-DHL with $(n-1)$ generation as long as the $p(b-1)$ internal spin variables within each basic unit are decimated.

The new Hamiltonian coupling constants are obtained by a formal set of renormalization equations given by

$$K^{(n-1)} = F_1(K^{(n)}, L^{(n)}, H^{(n)}),$$
$$H^{(n-1)} = F_3(K^{(n)}, L^{(n)}, H^{(n)}),$$
$$L^{(n-1)} = F_2(K^{(n)}, L^{(n)}, H^{(n)}),$$

where the functions F_1, F_2 and F_3, in general, are determined by the decimation process.

From this point, appropriate compact variables (*transmissivities*) introduced by Tsallis and Levy [24] were considered.

$$t_n = \frac{e^{q K^{(n)}} - 1}{e^{q K^{(n)}} + (q - 1)}, \quad (3)$$

$$u_n = \frac{e^{q H^{(n)}} - 1}{e^{q H^{(n)}} + (q - 1)}, \quad (4)$$

$$v_n = \frac{e^{q L^{(n)}} - 1}{e^{q L^{(n)}} + (q - 1)} \quad (5)$$

The equivalent decimation relations for the above variables (t, u, v), rather than (K, L, H) were obtained for the particular $(2, p)$-DHL family with the scaling factor $b = 2$. The choice of such hierarchical lattice family (scaling factor $b = 2$) is appropriated for studying the ferromagnetic model. To investigate the antiferromagnetic model, for instance, hierarchical lattices with odd scaling factor (odd b) should be required to preserve the ground-state configuration under renormalization. The deduction of the $b = 2$ decimation relations is presented in the

Appendix and leads formally to closed renormalization-group coupled equations given by

$$t_{n-1} = \frac{T(t_n, u_n) - 1}{T(t_n, u_n) + q - 1}, \tag{6}$$

$$u_{n-1} = \frac{U(t_n, u_n, v_n) - 1}{U(t_n, u_n, v_n) + q - 1}, \tag{7}$$

$$v_{n-1} = \frac{V(t_n, u_n, v_n) - 1}{V(t_n, u_n, v_n) + q - 1} \tag{8}$$

where

$$T(t, u) = \left[1 + \frac{qt^2(1 - u)}{(1 - t)(1 + t - 2tu)}\right]^p, \tag{9}$$

$$U(t, u, v) = \left[\frac{1 + (q - 1)u}{1 - u}\right]\left[\frac{N_2}{D}\right]^p, \tag{10}$$

$$V(t, u, v) = \left[\frac{N_3 D^2}{N_1 N_2^2}\right]^p \tag{11}$$

with

$$D = \frac{q(1 + t - 2tu)}{(1 - t)(1 - u)}, \tag{12}$$

$$N_1 = \frac{qt[2 + (q - 2)t]}{(1 - t)^2} + \frac{q}{1 - u}, \tag{13}$$

$$N_2 = q - 2 + \frac{1 + (q - 1)t}{1 - t} -$$
$$- \frac{[1 + (q - 1)t][1 + (q - 1)u][1 + (q - 1)v]}{(1 - t)(1 - u)(1 - v)}, \tag{14}$$

$$N_3 = q - 1 +$$
$$+ \frac{[1 + (q - 1)t]^2[1 + (q - 1)u][1 + (q - 1)v]^2}{(1 - t)^2(1 - u)(1 - v)^2}. \tag{15}$$

Previous results for the zero field pure q-state Potts model were re-obtained [13] when the limits of the zero field (u=0) and the zero correlation coupling (v=0) limits are imposed on the equations (6)-(8), that is

$$t_{n-1} = \frac{2t_n^2 + (q - 2)t_n^4}{1 + (q - 1)t_n^4}. $$

III. THE $q = 3$-STATE POTTS MODEL ON A (2,2)-DHL WITH FIELDS

The particular model with $q = 3$ and $b = p = 2$, was investigated in the present Section. Such model, which corresponds to the simplest one defined on a hierarchical lattice with integer fractal dimension $d_F > 1$, is suitable for comparision with the corresponding model defined on the square lattice in the framework of the real-space renormalization group approximation [20, 21]. Even though it presents a rich criticality with unusual characteristics on the critical surface. The study of other cases should follow straightforward the present calculation although with much heavy algebraic computational efforts.

A. The renormalization transformations

The renormalization equations for the 3-state Potts model defined on the (2, 2)-DHL, which has the fractal dimension $d_F = 2$, written in terms of the transmissivity variables, can be directly obtained from (6)- (15) by setting $q = 3$ and $p = 2$. After some algebraic rearrangements the following coupled renormalization equations are obtained:

$$t_{n-1} = \frac{T(t, u) - 1}{T(t, u) + 2}, \tag{16}$$

$$u_{n-1} = \frac{U(t, u, v) - 1}{U(t, u, v) + 2}, \tag{17}$$

$$v_{n-1} = \frac{V(t, u, v) - 1}{V(t, u, v) + 2} \tag{18}$$

where

$$T(t, u) = \frac{[1 + (2 - u)t^2 - 2tu]^2}{(1 - t)^2[1 + (1 - 2u)t]^2}, \tag{19}$$

$$U(t, u, v) = \frac{(1 + 2u)}{1 - u} \times$$
$$\times \frac{[1 + 2uv + t(1 + u + v + 3uv)]^2}{[1 + t(1 - 2u)]^2(1 - v)^2}, \tag{20}$$

$$V(t, u, v) = \frac{W(t, u, v)}{9[(1 - t)^2 + t(2 + t)(1 - u)]^2} \times$$
$$\times \frac{[1 + t(1 - 2u)]^4}{[1 + 2uv + t(1 + u + v + 3uv)]^4}, \tag{21}$$

$$W(t, u, v) = [2(1 - t)^2(1 - u)(1 - v)^2 +$$
$$+ (1 + 2t)^2(1 + 2u)(1 + 2v)^2]^2. \tag{22}$$

B. Renormalization Flow Diagram and Critical Points:

The 3D-renormalization parameter space in the variables (t, u, v) has the advantage of being confined to the region $(-0.5 \le t \le 1; \ -0.5 \le u \le 1; \ -0.5 \le v \le 1)$ the bounds corresponding to the $\pm\infty$ values of the respective former variables $(K, \ H, \ L)$. However the region $-0.5 \le t < 0$ corresponding to $K < 0$, although allowed by the variable transformation (3) must be excluded by the following conceptual reason: the $(2, p)$-DHL family (scaling factor two) is not suitable for studying the antiferromagnetic systems $(K < 0)$ since the RG decimation process does not preserve the antiferromagnetic ground state symmetry under renormalization. DHL with an odd scaling factor must be considered in order to study antiferromagnetic models within real space RG approaches.

Zero-field Potts model: $H = L = 0$ or line $u = v = 0$.			
t	u	v	description
0	0	0	Potts paramagnetic phase
$\frac{1}{2}$	0	0	Potts transition
1	0	0	Potts ferromagnetic phase
Negative infinite field $H = -\infty$ or plane $u = -0.5$			
t	u	v	description
0	$-\frac{1}{2}$	0	Ising paramagnetic phase
$0.442687\ldots$ ^{a}	$-\frac{1}{2}$	$-0.397513\ldots$	Ising like transition
1	$-\frac{1}{2}$	$-\frac{1}{2}$	Ising ferromagnetic phase
Zero temperature invariant plane: $T = 0$ or plane $t = 1$.			
t	u	v	description
1	1	$-\frac{1}{2}$	
1	0	0	Potts ferromagnetic phase
1	$-\frac{1}{2}$	$-\frac{1}{2}$	Ising ferromagnetic phase

^{a}Exact value$= -\left(\frac{4}{19}\right) + \frac{(17766 - 3078\sqrt{33})^{\frac{1}{3}}}{57} + \frac{[2(329 + 57\sqrt{33})]^{\frac{1}{3}}}{19}$

TABLE I: Fixed points of the renormalization flow diagram of the 3-state Potts Model under non-zero field. The t, u, v variables are defined by equations (3)-(5).
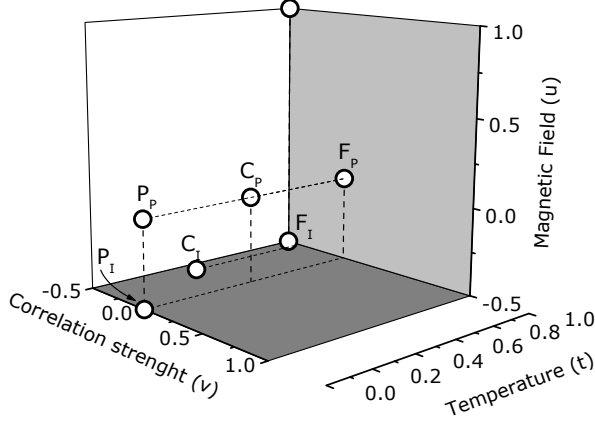

FIG. 2: 3D-plot displaying the fixed points described in Table I. P_P and F_P denote the fixed points associated with paramagnetic and ferromagnetic Potts phases, while C_P label the unstable fixed point. P_I, F_I and C_I indicate the corresponding fixed points associated with the Ising-like transition, respectively.

The fixed points of the corresponding recursion equations (16)-(18) are shown in Table I and sketched in Figure 2.

Three manifolds in the 3D cubic parameter space (t, u, v) can be distinguished:

(a) The zero-field and zero-correlation coupling Potts model fixed points located on the line $u = v = 0$, which contains the stable fixed points (0,0,0) and (1,0,0) associated with the paramagnetic and the ferromagnetic phases respectively, as well as the unstable fixed point (0.5,0,0) associated with the corresponding phase transition, the latter obtained by the exact solution of $t^* = 2t^{*2}/(1 + 2t^{*4})$.

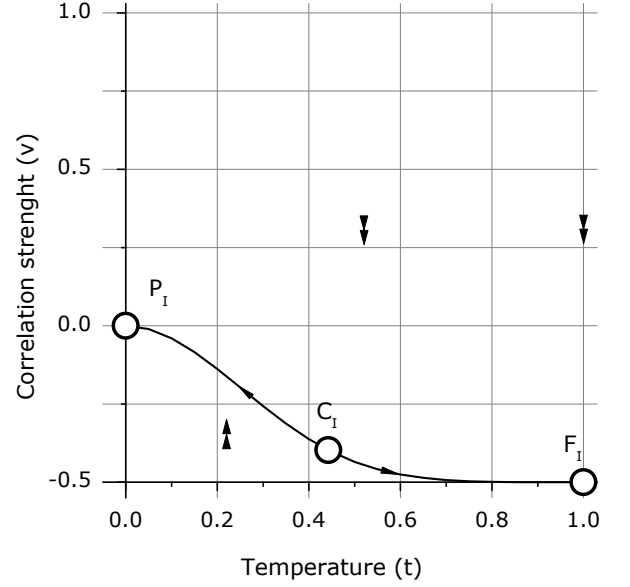(b) The $u = -1/2$ ($H \to -\infty$) plane, which corresponds

FIG. 3: Flow Diagram ($t \times v$) corresponding to an Ising like model in the invariant plane $u = -0.5$. The open circles mark the loci of the fixed points. P_I, F_I label the stable-fixed points of the paramagnetic (P) and ferromagnetic (F) phases while C_I labels the unstable-fixed point (critical point). The arrows indicate the flow direction, the double arrows signifying a direct jump to the flow line (solid line).

to the Ising model without magnetic field. The reduction of the number of states from $q = 3$ to $q = 2$ occurred because the 0-state became unreachable. For the general q-state Potts ferromagnetic model, whenever $u = -1/2$ ($H \to -\infty$) the 0-state becomes unreachable, leading to a ($q-1$)-state Potts model without magnetic field on such manifold. The renormalization flow in the $u = -0.5$ plane is depicted in Figure 3. The Ising-like paramagnetic $(0, -0.5, 0)$ and the ferromagnetic $(1,-0.5,-0.5)$ stable fixed points as well as the unstable phase transition points $(0.442687\ldots, -0.5, -0.397513\ldots)$ are found within this plane. Exact values may be obtained as indicated in Table I. The renormalization flow is confined in the plane $u = -0.5$ and given by

$$t' = \frac{3t^2(2+t)^2}{4 + 8t - 4t^3 + 19t^4} \qquad (23)$$

$$v' = \frac{2A - B}{A + B} \qquad (24)$$

with

$A = 16[(1 + 2t)(1 - t)]^4$

$B = [2(1 - t)^2 + 3(2 + t)]^2(2 + t)^4.$

(c) The zero temperature invariant plane ($t = 1$), which contains both the Potts and the Ising fixed points, $(1, 0, 0)$ and $(1, -0.5, -0.5)$ respectively, is displayed in the Figure 4.
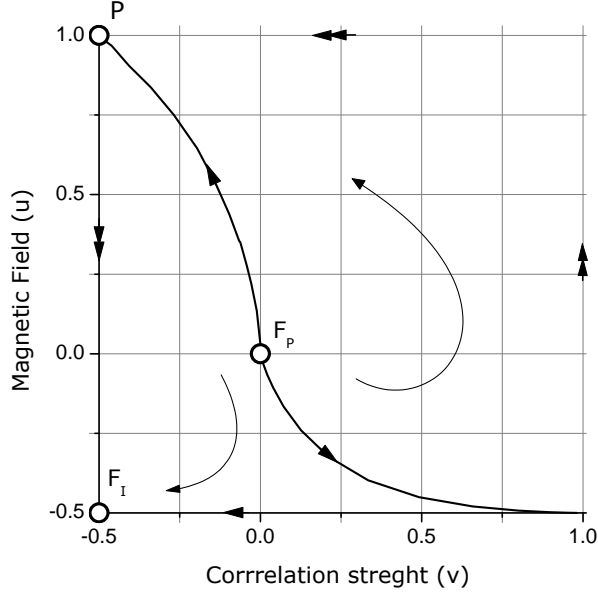
FIG. 4: Flow Diagram $(v \times u)$ corresponding to the zero temperature invariant plane $t = 1$. Open symbols mark the fixed points, F_P labels the stable fixed point of the Potts ferromagnetic phase, F_I labels the stable fixed point of the Ising ferromagnetic phase, while P indicates the stable fixed-point associated paramagnetic phase (infinite H field). The arrows indicate the flow direction, the double arrows signifying a direct jump to the end point. The solid plot corresponds to the intersection of the critical surface with the $t = 1$ plane. Note the singularity in the derivative of the plot at the point F_P (0,0).

IV. DISCUSSION AND CONCLUSIONS

The critical properties of the q-state ferromagnetic Potts model defined on a fractal hierarchical lattice in the presence of a magnetic field, were investigated through a real-space renormalization group (RG) technique. The inclusion of a new interaction term in the Hamiltonian that boosted the coupling of pairs of spins in the 0th state (the state chosen to be singled out by the external field) led to a closed – and exact – coupled RG transformation. The full renormalization flow diagram in an appropriate compact parameter space is analyzed for the particular model with $q = 3$ Potts states, defined on the diamond hierarchical lattice with a fractal dimension $d_F = 2$. The inclusion of such a correlation coupling breaks the symmetry for the pair correlation with respect to the Potts state singled out by the external field. Whenever $q \geq 3$ pairs of spins with states parallel to the external field have different energies with respect to all pairs of parallel spins in other states. Note that such a correlation coupling has no relevance for the Ising model $(q = 2)$ leading only to rescale the energy of the ferromagnetic pair interaction.

New and interesting features are found within the global phase diagram:
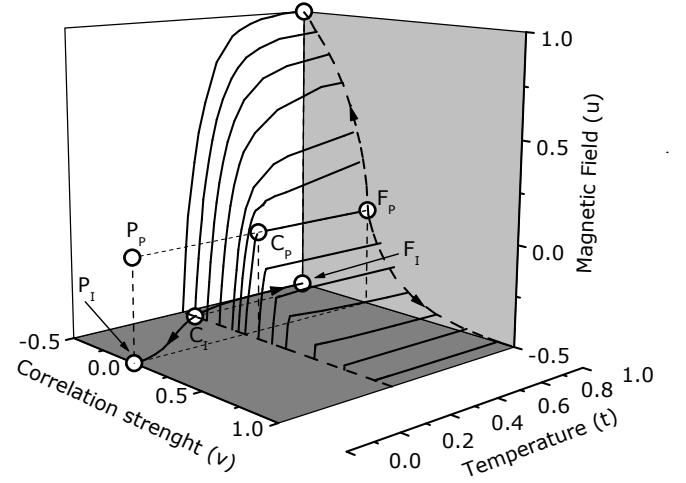

FIG. 5: Profiles of intersections of the critical surface with the planes $v = -0.5$, -0.395, -0.3, -0.2, -0.1, -0.05, 0.0, 0.1, 0.2, 0.3, 0.5, 0.7 and 0.9 (solid lines), from top to bottom. Open circles mark the loci of the fixed points, F_P, P_P, C_P label ferromagnetic, paramagnetic and Potts transition fixed points, while F_I, P_I and C_I indicate the corresponding Ising like fixed points at the plane $u = -0.5$.

(a) A dimensional reduction of the order parameter at a strong (infinite) reverse magnetic field, which depopulates the 0th Potts state: within the plane $(u = -1/2)$ the system is reduced to a $(q - 1)$-Potts like model. For the studied case $(q = 3)$ the reduced model corresponds to an effective Ising ferromagnet. Within the 3D-parameter space flow diagram the stable fixed point associated with the Ising condensed phase is located at zero temperature $(t = 1)$ and infinite reverse magnetic field $(u = -1/2)$.

(b) The basin of attraction of such a point is separate from that of the paramagnetic phase by a complex boundary surface, which contains the locus of the zero field q-state Potts phase transition. Both the unstable and the stable fixed points associated with the Potts phase belongs to the boundary surface along the line $(u = v = 0, t \geq 1/2)$

(c) Along such a line, the boundary surface has a sharp cusp. Other lines with a sharp cusp also appears in the $v \geq 0$ portion of the surface.

Figure 5 displays several plots of the intersection of the critical surface with planes (t, u) for chosen values of v, showing all the above-mentioned features. Figure 6 exhibits in appropriated scale the details of sharp cusps appearing in the plots of the intersection of the critical surface with the planes $v = 0.7$ and $v = 0.9$. Whether such features are present in the q-state Potts model defined in other kinds of lattices deserves further investigations.

The eigenvalues and eigenvectors associated with the Potts and Ising unstable fixed points were calculated. Table II displays such result. The α and ν critical exponents
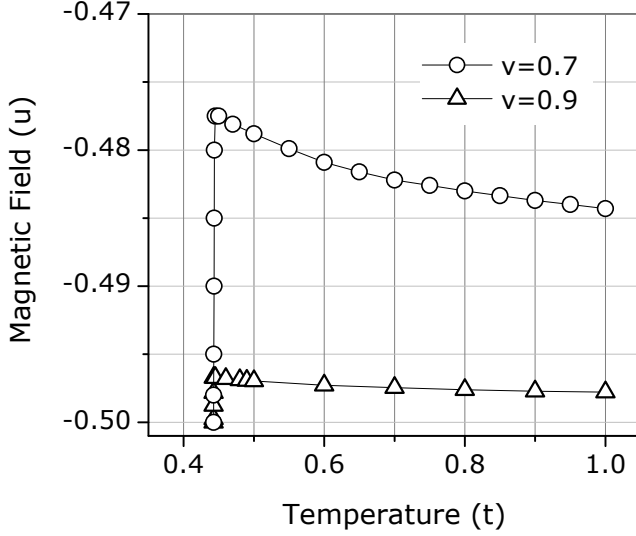
FIG. 6: Details of intersections of the critical surface with the planes $v = 0.7$ and $v = 0.9$, from top to bottom. The vertical scale was expanded to emphasize the cusps for lower values of t.

Potts transition point: (0.5, 0, 0)		
$\lambda_t = 2.26938 \rightarrow \vec{v}_t = (-0.18388, 0.813562, 0.551639)$		
$\lambda_u = 1.77777 \rightarrow \vec{v}_u = (1, 0, 0)$		
$\lambda_v = 1.17506 \rightarrow \vec{v}_v = (-0.159264, -0.863917, 0.477792)$		
Ising transition point: (0.442687, -0.5, -0.397513)		
$\lambda_t = 1.678573 \rightarrow \vec{v}_t = (0.912186, 0, -0.409776)$		
$\lambda_u = 0.419643 \rightarrow \vec{v}_u = (0.0499605, 0.99851, -0.0219619)$		
$\lambda_v = 0 \rightarrow \vec{v}_v = (0, 0, 1)$		

TABLE II: Eigenvalues λ's and the corresponding eigenvectors $\vec{v}$'s associated with the transition Potts and Ising fixed points.

at the fixed points associated with the Potts and Ising transitions, previously reported in the literature[13] were recovered. The general nature of the transitions studied in this work (for $q = 3$) is found to be of a second order. Other authors [14] have claimed that, at higher values of q, the inherent inhomogeneity of the model might induce a change to first-order transition, which provides another question for further investigation.

In closing, to the current knowledge of the authors, the study reported throughout this paper is one of a few exact renormalization group approaches to the Potts model under an external magnetic field in a growing body of literature on the subject.

APPENDIX: EXACT RENORMALIZATION EQUATIONS FOR THE q-STATE POTTS MODEL ON $(2, p)$-DHL

The Hamiltonian for a single basic unit with one internal site $b = 2$ can be written as:

$$-\beta\mathcal{H} = qK \sum_{i=1}^{p} (\delta_{\sigma_i \mu_1} + \delta_{\sigma_i , \mu_2}) + qL \sum_{i=1}^{p} (\delta_{\sigma_i 0} \delta_{\mu_1 0} +$$
$$+ \delta_{\sigma_i 0} \delta_{\mu_2 0}) + qH \sum_{i=1}^{p} [\delta_{\sigma_i 0} + \delta_{\mu_1 0} + \delta_{\mu_2 0}], \quad \text{(A.1)}$$

which can be rewritten for latter convenience as

$$-\beta\mathcal{H} = \sum_{i=1}^{p} \mathcal{H}_i$$

with

$$\mathcal{H}_i = qK(\delta_{\sigma_i \mu_1} + \delta_{\sigma_i \mu_2}) + qL(\delta_{\sigma_i 0} \delta_{\mu_1 0} + \delta_{\sigma_i 0} \delta_{\mu_2 0}) +$$
$$+ qH\delta_{\sigma_i 0} + \frac{q}{p}H\delta_{\mu_1 0} + \frac{q}{p}H\delta_{\mu_2 0}. \quad \text{(A.2)}$$

The corresponding restricted partition function for fixed configuration of the root sites:

$$z(\mu_1, \mu_2) = Tr_{\{\sigma_i\}} e^{-\beta\mathcal{H}} = Tr_{\{\sigma_i\}} \prod_{i=1}^{p} e^{\mathcal{H}_i} =$$
$$= \{Tr_{\sigma_i} e^{\mathcal{H}_i}\}^p = \left\{ \sum_{\sigma=0}^{q-1} \exp\left[qK(\delta_{\sigma \mu_1} + \delta_{\sigma \mu_2}) + \right.\right.$$
$$+ qL(\delta_{\sigma 0}\delta_{\mu_1 0} + \delta_{\sigma 0}\delta_{\mu_2,0}) + \frac{q}{p}H(p\,\delta_{\sigma 0} +$$
$$\left.\left. + \delta_{\mu_1 0} + \delta_{\mu_2 0})\right]\right\}^p. \quad \text{(A.3)}$$

The renormalized bond Hamiltonian is defined by

$$-\beta\mathcal{H}' = qK'\delta_{\mu_1 \mu_2} + qL'\delta_{\mu_1 0}\delta_{\mu_2 0} + qH'(\delta_{\mu_1 0} + \delta_{\mu_2 0}), \quad \text{(A.4)}$$

while the renormalized restricted partition function results as:

$$z'(\mu_1, \mu_2) = e^{-\beta\mathcal{H}'} = \exp\left[qK'\delta_{\mu_1 \mu_2} + \right.$$
$$\left. + qL'\delta_{\mu_1 0}\delta_{\mu_2 0} + qH'(\delta_{\mu_1 0} + \delta_{\mu_2 0})\right]. \quad \text{(A.5)}$$

The partition functions Z' and Z are compared to obtain the renormalization equations for the coupling constants K', L', H' as a function of K, L, H, that is

$$Z \equiv A Z',$$
$$\sum_{\{\mu_1, \mu_2\}} z(\mu_1, \mu_2) = A \sum_{\{\mu_1, \mu_2\}} z'(\mu_1, \mu_2), \quad \text{(A.6)}$$

where A is a constant to be determined as a function of K, L, H, which can be used to fix the origin of the energy scale. The expansions of the configurations of the root sites in the sums of equation (A.6) are performed and four kinds of possible configurations may be distinguished, namely

1. $\mu_1 = \mu_2 = 0$, one configuration.

2. $\mu_1 = 0$ and $\mu_2 \neq 0$ and *vice- versa*, two configurations.

3. $\mu_1 = \mu_2 \neq 0$, leading to $(q-1)$ configurations.

4. $\mu_1 \neq \mu_2$, $\mu_1\mu_2 \neq 0$, leading to $(q-1)(q-2)$ configurations.

The contributions for the partition functions are equal by symmetry for each case. The comparison term by term led to the following relations:

$$A\,e^{qK'+qL'+2qH'} = e^{2qH}(e^{2qK+2qL+qH}+q-1)^p, \tag{A.7}$$

$$A\,e^{qH'} = e^{qH}(e^{qK}+e^{2qH+qL}+q-2)^p, \tag{A.8}$$

$$A\,e^{qK'} = (e^{qH}+e^{2qK}+q-2)^p, \tag{A.9}$$

$$A = (e^{qH}+2e^{qK}+q-3)^p, \tag{A.10}$$

which can be solved for the coupling constants K', L', H' leading to

$$e^{qK'} = \left(\frac{e^{2qK}+e^{qH}+q-2}{2e^{qK}+e^{qH}+q-3}\right)^p, \tag{A.11}$$

$$e^{qH'} = e^{qH}\left(\frac{e^{qK}+e^{qK+qL+qH}+q-2}{2e^{qK}+e^{qH}+q-3}\right)^p, \tag{A.12}$$

$$e^{qL'} = \left[\frac{(e^{2qK+2qL+qH}+q-1)}{(e^{2qK}+e^{qH}+q-2)}\times\right.$$
$$\left.\times\frac{(2e^{qK}+e^{qH}+q-3)^2}{(e^{qK}+e^{qK+qL+qH}+q-2)^2}\right]^p. \tag{A.13}$$

New appropriated compact variables t, u, v, defined in the interval $(-1/(q-1), 1)$ instead of the K, H, L defined in $(-\infty, \infty)$, are considered respectively by

$$t = \frac{e^{qK}-1}{e^{qK}+q-1}, \tag{A.14}$$

$$u = \frac{e^{qH}-1}{e^{qH}+q-1}, \tag{A.15}$$

$$v = \frac{e^{qL}-1}{e^{qL}+q-1}. \tag{A.16}$$

The inverse relations give

$$e^{qK} = \frac{1+(q-1)t}{1-t}, \tag{A.17}$$

$$e^{qH} = \frac{1+(q-1)u}{1-u}, \tag{A.18}$$

$$e^{qL} = \frac{1+(q-1)v}{1-v}. \tag{A.19}$$

Equations (A.17)-(A.19) are substituted in (A.11)-(A.13), which by it turns are substituted in the equations (A.14)-(A.16) after changing $(t \to t')$. The result is

$$t' = \frac{T(t,u)-1}{T(t,u)+q-1}, \tag{A.20}$$

$$u' = \frac{U(t,u,v)-1}{U(t,u,v)+q-1}, \tag{A.21}$$

$$v' = \frac{V(t,u,v)-1}{V(t,u,v)+q-1}, \tag{A.22}$$

where

$$T(t,u) = \left[1+\frac{qt^2(1-u)}{(1-t)(1+t-2tu)}\right]^p, \tag{A.23}$$

$$U(t,u,v) = \left[\frac{1+(q-1)u}{1-u}\right]\left[\frac{N_2}{D}\right]^p, \tag{A.24}$$

$$V(t,u,v) = \left[\frac{N_3\,D^2}{N_1\,N_2^2}\right]^p \tag{A.25}$$

with

$$D = \frac{q(1+t-2tu)}{(1-t)(1-u)}, \tag{A.26}$$

$$N_1 = \frac{qt[2+(q-2)t]}{(1-t)^2}+\frac{q}{1-u}, \tag{A.27}$$

$$N_2 = q-2+\frac{1+(q-1)t}{1-t}-$$
$$-\frac{[1+(q-1)t][1+(q-1)u][1+(q-1)v]}{(1-t)(1-u)(1-v)}, \tag{A.28}$$

$$N_3 = q-1+$$
$$+\frac{[1+(q-1)t]^2[1+(q-1)u][1+(q-1)v]^2}{(1-t)^2(1-u)(1-v)^2}. \tag{A.29}$$

Equations (A.20)-(A.29) give the complete set of the renormalization equations for the q-state Potts Model defined on a general diamond hierarchical lattice with scaling factor $b = 2$ and fractal dimension $d_F = 1 + \log p/\log 2$, whose Hamiltonian is given by equation (2).

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